Multi-Robot Systems: A Control Perspective

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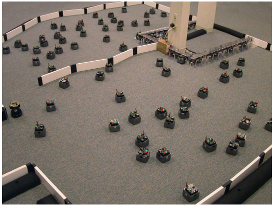
Outline

What is a Multi-Robot System?

Formally, a collection of two or more autonomous mobile robots working together is termed as team or society of mobile robots.



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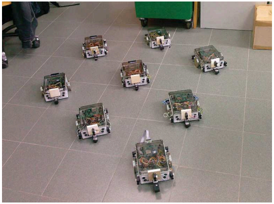


"Roma Tre" University Robotics and Sensor Fusion Laboratory

Zaragoza University Robotics, Perception and Real-Time Group

Reference:

- L. E. Parker. "Multiple Mobile Robot Systems". In: *Springer Handbook of Robotics*. Ed. by B. Siciliano and O. Khatib. Springer Handbooks, 2008. Chap. 40

Why would I want it?

Multi-robot systems can be of interest for several reasons:

- the task complexity is too high for a single robot,
- the task is inherently distributed,
- the design of several resource-bounded robots is much easier than a single powerful one,
- multiple robots can solve problems faster using parallelism,
- the introduction of multiple robots increases robustness through redundancy.

How can a taxonomy be drawn?

A taxonomy of Multi-Robot Systems can be derived considering:

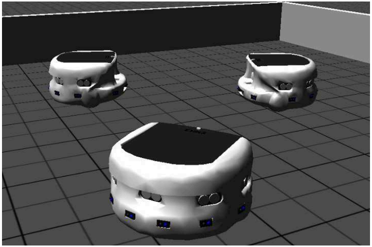
- Nature of the Team
- Control Architecture
- Communication Scheme

Introduction

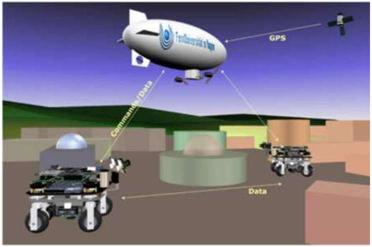
Nature of the Team

Homogeneous vs Heterogeneous

A team of robots might consist of identical units or different units.



Homogeneous Team



Heterogeneous Team

Homogeneous Team of Robots

Typical of swarms robotics. Classical properties are:

- Each unit has the same capabilities
- High level of redundancy
- Little ability to solve meaningful tasks for each robot
- Higher ability to solve task by teaming-up (superadditivity)

Heterogenous Team of Robots I

Complex applications of large-scale robot teams may require:

- the simultaneous use of multiple types of sensors,
- the simultaneous use of multiple types of robots.

For instance:

- Some sensors may be too expensive to duplicate across all robots on the team,
- Some robots may need to be scaled to smaller sizes, which will limit their payloads.

Heterogenous Team of Robots II

- Heterogeneity can offer economic benefits:

  It might be cheaper to distribute different capabilities across multiple team members rather than to build many copies of monolithic robots.

- Heterogeneity can offer engineering benefits:

  It may simply be too difficult to design individual robots that incorporate all of the sensing, computational, and effector requirements of a given application

Introduction

Control Architecture

Overview

The design of the overall control architecture for the multi-robot team has a significant impact on:

- robustness
- scalability

Different kind of architectures can be considered:

- Centralized
- Hierarchical
- Distributed
- Hybrid

Centralized

Centralized architectures assume the coordination of the entire team to be carried out by a single point of control.

Advantages:

- Control system design,
- Overall achievable performances.

Drawbacks:

- Communication complexity to achieve real-time control,
- Vulnerability to single point of failure.

Hierarchical

Hierarchical architectures assume that each robot oversees the actions of a relatively small group of other robots, each of which in turn oversees yet another group of robots, and so forth, down to the lowest robot, which simply executes its part of the task.

Advantages:

- Scalability w.r.t. centralized approaches,

Drawbacks:

- Fragility with respect to certain failures, e.g., robots “high” in the control tree.

Decentralized

Decentralized architectures require robots to take actions based only on knowledge local to their situation.

Advantages:

- Robustness to failure,
- Control design.

Drawbacks:

- Achieving global coherency,
- Overall achievable performances.

Hybrid

Hybrid control architectures combine local control with higher-level control approaches to enhance both robot autonomy and explicit coordination.

Advantages:

- Overall achievable performances,
- Robustness to failure.

Drawbacks:

- Overall system design,
- Scalability w.r.t. the tasks.

Introduction

Communication Scheme

Overview

A fundamental assumption in multi-robot systems research is that:

"globally coherent and efficient solutions can be achieved through the interaction of robots lacking complete global information"

$\Downarrow$

Achieving these globally coherent solutions typically requires robots to obtain information about their teammates states or actions.

Taxonomy

Information can be gathered in a number of ways. The three most common techniques are:

- Implicit communication through the world
- Passive action recognition
- Explicit (intentional) communication

Implicit communication through the world

Robots sense the effects of teammate's actions through their effects on the world (Stigmergy).

Advantages

- Appealing because of its simplicity and its lack of dependence upon explicit communications channels and protocols.

Drawbacks

- It is limited by the extent to which a robot's perception of the world reflects the salient states of the mission the robot team must accomplish.

Passive action recognition

Robots use sensors to directly observe the actions of their teammates.

Advantages

- Appealing because it does not depend upon a limited bandwidth, fallible communication mechanism.

Drawbacks

- It is limited by the degree to which a robot can successfully interpret its sensory information, as well as the difficulty of analyzing the actions of robot team members.

Explicit (intentional) communication

Robots directly and intentionally communicate relevant information through some active means, such as radio.

Advantages

- Appealing because of its directness and the easiness with which robots can become aware of the actions and/or goals of its teammates.

Drawbacks

- It is limited in terms of fault tolerance and reliability, because it typically depends upon a noisy, limited-bandwidth communications channel.

How to pick one out?

Selecting the appropriate means of communication is a design choice dependent upon the tasks to be achieved

Observations:

- Costs and benefits of alternative communications approaches must be carefully analyzed to determine the method that can reliably achieve the required level of system performance.
- Researchers generally agree that communication can have a strong positive impact on the performance of the team.

How much should we communicate?

A nonlinear relationship exists between the amount of information exchanged and its impact on the performance of the team.

- Even a small amount of information can have a significant impact on the team
- More information might not continue to improve performance, e.g., bandwidth overload with no application benefit

The challenge is to discover the optimal pieces of information to exchange that yield these performance improvements without saturating the communications bandwidth

Introduction

Research Interests

What are the related Research Problems?

Major research areas concerning Multi-Robot Systems are:

- Localization, Mapping and Exploration
- Task Allocation and Sequencing
- Distributed Coordination

References:

- Sebastian Thrun and John J. Leonard. "Simultaneous Localization and Mapping". In: *Springer Handbook of Robotics*. Ed. by B. Siciliano and O. Khatib. Springer Handbooks, 2008. Chap. 37
- P. Brass, F. Cabrera-Mora, A. Gasparri, and Xiao Jizhong. "Multirobot Tree and Graph Exploration". In: *IEEE Transactions on Robotics* 27.4 (2011), pp. 707–717
- M.B. Dias, Robert Zlot, N. Kalra, and A. Stentz. "Market-Based Multirobot Coordination: A Survey and Analysis". In: *Proceedings of the IEEE* 94.7 (2006), pp. 1257–1270

Algebraic Graph Theory

Preliminaries

Quadratic Forms

Definition

In mathematics, a quadratic form is a homogeneous polynomial of degree two in a number of variables:

$$Q(x) = a\,x_1^2 + b\,x_2^2 + cx_1\,x_2 \qquad (1)$$

A quadratic form can always be expressed by using a vector $x \in \mathbb{R}^n$ and a symmetric matrix $A \in \mathbb{R}^{n\times n}$ as follows:

$$Q(x) = x^T A\,x \qquad (2)$$

Positive Definiteness I

- An $n \times n$ real symmetric matrix M is positive definite if:

$$z^T M z > 0, \qquad z \in \mathbb{R}^n, z \neq \mathbf{0}, \tag{3}$$

where z^T denotes the transpose of z.

- An $n \times n$ real symmetric matrix M is positive semi-definite if:

$$z^T M z \geq 0, \qquad z \in \mathbb{R}^n, \tag{4}$$

where z^T denotes the transpose of z.

Positive Definiteness II

How can we check the positiveness of a real symmetric matrix A?

By looking at its spectrum!!!

A real symmetric matrix A is positive definite iff all its eigenvalues are positive, namely:

$$A > 0 \iff \lambda_i > 0, \quad \forall\, \lambda_i \sigma(A)$$

Observations:
A real symmetric matrix A can always be diagonalized by means of an orthogonal matrix Q, i.e. a matrix such that $Q^{-1} = Q^T$.

Positive Definiteness III

A real matrix M may have the property that $x^T Ax > 0$ for all nonzero real vectors x without being symmetric.

How can we check the positiveness of a real matrix A?

By looking at the spectrum of its symmetric part!!!

Definition:
The symmetric part A^+ of a matrix $A \in \mathbb{R}^{n\times n}$ can be defined as:

$$A^+ = \frac{A + A^T}{2} \tag{5}$$

Gerschgorin Circle Theorem I

The Gershgorin circle theorem can be used to provide a bound for the spectrum of a square complex matrix A.

Let the *Gershgorin disc D_i* associated with the i-th row be defined as:

$$D_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \le R_i \right\}, \quad R_i = \sum_{j \neq i} |a_{ij}| \tag{6}$$

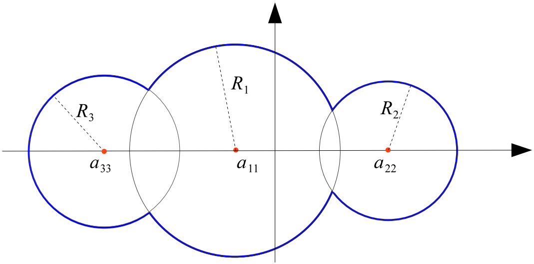
with R_i the sum of the absolute values of the off-diagonal entries in the i-th row.

Gerschgorin Circle Theorem II

The Gershgorin circle theorem states that every eigenvalue of the complex matrix A lies within the union of the Gershgorin discs D_i, that is:

$$\lambda_i \in \bigcup_{i=1}^{n} D_i, \ \forall \ \lambda_i \in \sigma(A),$$

where $\sigma(\cdot)$ is the set of eigenvalues of a matrix.

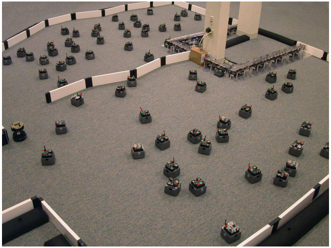


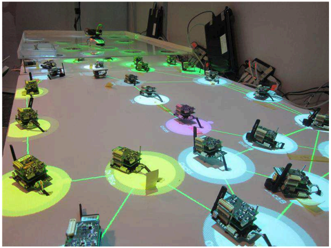
Algebraic Graph Theory

Graph Modeling

Multi-Agent System Modeling I

Multi-agent systems (MASs) represent an ideal abstraction of actual networks of mobile robots or sensor nodes that are envisioned to perform the most various kind of tasks.





Multi-Agent System Modeling II

The interaction among agents is capture by the network topology which can be described by means of a graph $\mathcal{G} = \{V, E\}$.

$\Downarrow$

Tools coming from the Algebraic Graph Theory can be used to formally describe the interaction among the agents.

Definitions II

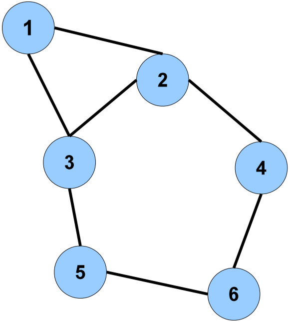
Consider a MAS described by an undirected graph $\mathcal{G} = (V, E)$.

A matrix based representation can be obtained by introducing:

- **Adjacency Matrix (A)**: is a binary matrix used to represent which vertices (or nodes) of a graph are adjacent to which other vertices,
- **Degree Matrix (D)**: is a diagonal matrix which contains information about the degree of each vertex
- **Laplacian Matrix (L)**: is a matrix representation of a graph. It can be used to find many properties of the graph

Definitions III

Adjacency Matrix

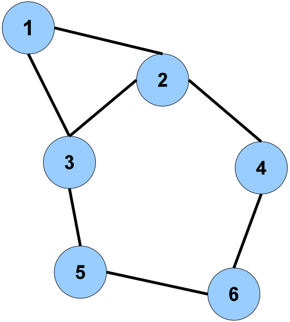


$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

This matrix is symmetric if and only if the graph is undirected.

Definitions IV

Degree Matrix

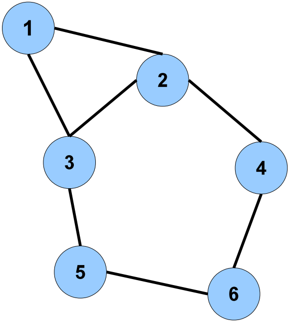


$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

In the case of a directed graph, there are two different degree matrices: 1) in-degree matrix, 2) out-degree matrix.

Definitions V

Laplacian Matrix



$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

The Laplacian matrix can be defined by starting from the adjacency matrix (A) and degree matrix (D) as $L = D - A$.

Laplacian Matrix: Properties I

Consider a $n \times n$ Laplacian matrix L of an undirected graph:

- L is a symmetric positive semi-definite matrix,
- L has $\mathrm{rank}\,(L) = n - c$, with c the number of connected components,

Notation:
In the following, the eigenvalues of the Laplacian matrix L of a connected graph will be denoted as follows:

$$0 = \lambda_1 < \lambda_2 \leq \ \ldots \ \leq \lambda_m \leq \min\{2\,\Delta, n\}$$

Laplacian Matrix: Properties II

If the graph $\mathcal{G}$ is connected, the Laplacian matrix L has:

- $\lambda_1 = 0$ with $v_1 = \mathbf{1}$
- $\mathbf{1}^T L = \mathbf{0}^T$ and $L\,\mathbf{1} = \mathbf{0}$
- λ_2 is the algebraic connectivity
- $\lambda_{max} \leq 2\,\Delta$, with Δ the maximum degree.
- $\lambda_{max} \leq n$, with n the number of vertexes.

Lyapunov Theory

Equilibrium Points I

Consider a non-linear system:

$$\dot{x} = f(x), \quad f : \mathbb{R}^n \to \mathbb{R}^n \tag{7}$$

A point x_e is said to be an equilibrium point if:

$$f(x_e) = 0 \tag{8}$$

Observations

- A linear system has either one equilibrium point (the origin) or a subspace of equilibrium points
- A nonlinear system might have several (isolated) equilibrium points

Reference:

- Hassan K. Khalil. *Nonlinear Systems (3rd Edition)*. 3rd ed. Prentice Hall, Dec. 2001. ISBN: 0130673897

Equilibrium Points II

Let us consider a linear dynamical system:

$$\dot{x} = A\,x \tag{9}$$

with a state-transition matrix A defined as:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \tag{10}$$

The subspace of equilibrium points is given by:

$$A\,x_e = 0 \quad \text{that is} \quad x_e \in \mathcal{N}(A) \tag{11}$$

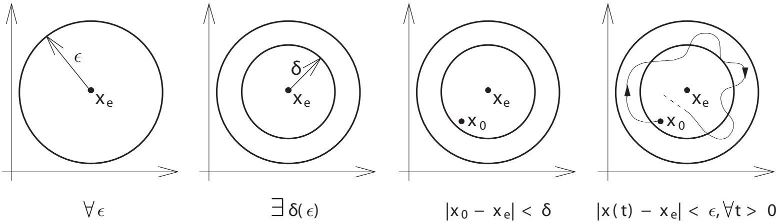
that is:

$$x_e = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \alpha \in \mathcal{R} \tag{12}$$

Stability of Equilibrium Points

Suppose x_e is an equilibrium point, then x_e is stable if:

$$\forall \epsilon, \quad \exists \delta(\epsilon) : \|x(0) - x_e\| \leq \delta(\epsilon) \Rightarrow \|x(t) - x_e\| \leq \epsilon, \forall t > 0 \quad (13)$$



An equilibrium point x_e is stable if the trajectory $x(t)$ can be kept arbitrarily close to it over time by opportunely choosing the initial conditions $x(0)$. The equilibrium is said unstable otherwise.

Stability of Equilibrium Points

Suppose x_e is an equilibrium, then x_e is asymptotically stable if:

i) x_e is stable

ii)
$$\exists\, \delta_a \; : \; \|x(0) - x_e\| < \delta_a \;\Rightarrow\; \lim_{t \to \infty} \|x(t) - x_e\| = 0 \qquad (14)$$

with δ_a the radius of the attraction domain.

Observations:

- Asymptotic stability is a local concept (it depends upon the attraction domain)
- The condition ii) does not imply the condition i)
- The asymptotic stability is global if it holds for any initial condition, i.e., the attraction domain is $\mathbb{R}^n$ and there is a unique x_e

Lyapunov Function I

Consider a continuously differentiable function $V : S(x_e, r) \rightarrow \mathbb{R}$:

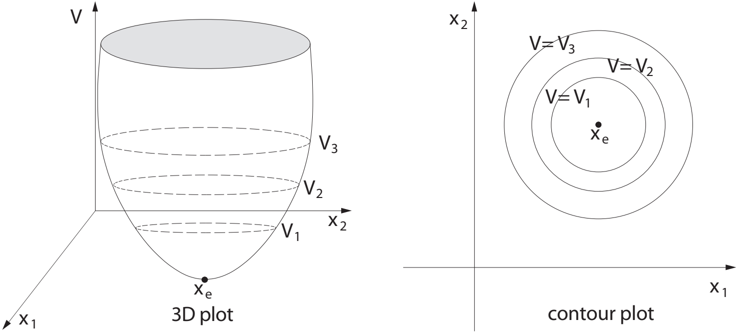
- $V(x)$ is positive definite in $S(x_e, r)$ if:
  - a) $V(x_e) = 0$
  - b) $V(x) > 0, \quad \forall x \in S(x_e, r)$
- $V(x)$ is positive semi-definite in $S(x_e, r)$ if:
  - a) $V(x_e) = 0$
  - b) $V(x) \geq 0, \quad \forall x \in S(x_e, r)$

Observations:

- $V(x)$ is negative (semi) definite if $-V(x)$ is positive (semi) definite
- A quadratic form can represent a candidate Lyapunov function

Lyapunov Function II

Example of a positive definite Lyapunov function $V : \mathbb{R}^2 \rightarrow \mathbb{R}$:



$$V(x) = x^T x = x_1^2 + x_2^2, \quad x \in \mathbb{R}^2$$

Lyapunov Function Derivative

Compute the Lyapunov time derivative $\dot{V} : \mathbb{R}^n \to \mathbb{R}$ as:

$$\dot{V}(t) = \frac{\mathrm{d}\, V(x)}{\mathrm{d}\, t} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial t} = \frac{\partial V}{\partial x} \dot{x} \tag{15}$$

By substituting the dynamics of the system $\dot{x} = f(x)$ it follows:

$$\dot{V}(t) = \frac{\partial V}{\partial x} \dot{x} = \frac{\partial V}{\partial x} f(x) = \dot{V}(x) \tag{16}$$

$\dot{V}(x)$ is the derivative of $V(x)$ along the trajectories of the system $\dot{x} = f(x)$.

Observations:

- If $\dot{V}(x) < 0$, then $V(x)$ decreases along the trajectories of the system.

Lyapunov Stability Idea

- Consider the function $V(x)$ to be an energy-like function
- If the energy of the system is constantly dissipating ($\dot{V} < 0$)
- The system must tend towards an equilibrium point s.t. $\dot{V} = 0$

$$\Downarrow$$

- The stability of the system can be studied "simply" by analyzing a scalar function $V(x)$
- A closed form for the system trajectories is not required to understand the "qualitative" behavior of the system

Lyapunov Stability

Theorem: Let $x_e \in \mathbb{R}^n$ be an equilibrium point for the system (8). Then x_e is stable if there exits a Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that:

i) $V(x)$ is positive definite in $S(x_e, r)$

ii) $\dot{V}(x)$ is negative semi definite in $S(x_e, r)$

Lyapunov Asymptotic Stability

Theorem: Let $x_e \in \mathbb{R}^n$ be an equilibrium point for the system (8). Then x_e is asymptotically stable if there exits a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ such that:

i) $V(x)$ is positive definite in $S(x_e, r)$

ii) $\dot{V}(x)$ is negative definite in $S(x_e, r)$

Lyapunov Global Asymptotic Stability

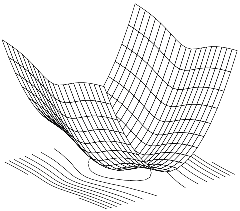
Theorem: Let $x_e \in \mathbb{R}^n$ be an equilibrium point for the system (8). Then x_e is globally asymptotically stable if there exits a Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that:

i) $V(x)$ is positive definite in $\mathbb{R}^n$

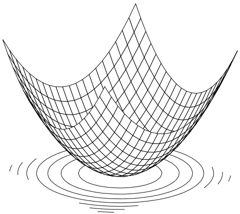
ii) $\dot{V}(x)$ is negative definite in $\mathbb{R}^n$

Iii) $V(x)$ is radially unbounded, that is $\lim_{\|x-x_e\|\rightarrow\infty} V(x) = \infty$

Radially bounded $V = \frac{x_1^2}{1+x_1^2} + x_2^2$



Radially unbounded $V = x_1^2 + x_2^2$



Consensus Problem

Problem Definition I

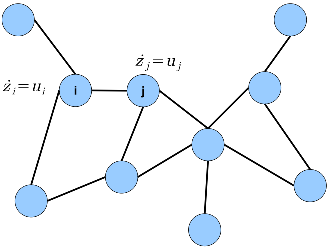
In networks of agents (or dynamic systems), consensus means to reach an agreement regarding a certain quantity of interest that depends on the state of all agents.

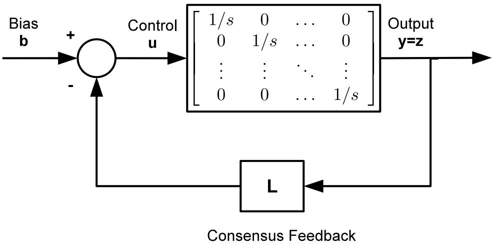
A consensus algorithm (or protocol) is a local interaction rule that specifies the information to be exchanged between neighboring agents on the network to reach such an agreement.

Reference:

- R. Olfati-Saber, J.A. Fax, and R.M. Murray. "Consensus and Cooperation in Networked Multi-Agent Systems". In: *Proceedings of the IEEE* 95.1 (2007), pp. 215–233

Problem Definition II





Two equivalent forms of consensus algorithms:

(a) a network of integrator agents in which agent i receives the state z_j of its neighbor, agent j, if there is a link e_{ij} connecting the two nodes;

(b) the block diagram for a network of interconnected dynamic systems all with identical transfer functions $P(s) = 1/s$. The collective networked system has the Laplacian matrix in the feedback loop.

Consensus Problem

Continuous Time

Problem Stament I

Assume each agent i has a single integrator dynamics:

$$\dot{z}_i = u_i \tag{17}$$

Consider the following control law u_i for an agent i:

$$\dot{z}_i(t) = \sum_{j \in \mathcal{N}_i} \left(z_j(t) - z_i(t) \right) \tag{18}$$

with $\mathcal{N}_i = \{i \in V \ : \ e_{ij} \in E\}$ the neighborhood of agent i.

Problem Stament II

The collective dynamics of the group of agents is:

$$\dot{z}(t) = -L\,z(t), \tag{19}$$

with:

- $z = [z_1, \ldots, z_n]^T$ the stacked vector of agents state
- L the graph Laplacian of the network whose elements are:

$$l_{ij} = \begin{cases} -1 & j \in \mathcal{N}_i, \\ |\mathcal{N}_i| & j = i, \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

with $\mathcal{N}_i$ denotes the number of neighbors of node i.

Main Result

Theorem: Consider the collective dynamics given in eq. (19), if the graph is connected a consensus is asymptotically achieved to the average of the initial state $\{z_i(0)\}_{i=1}^n$ of all nodes, that is:

$$z(\infty) = \alpha \cdot \mathbf{1}, \tag{21}$$

where

$$\alpha = \frac{1}{n} \sum_{i=1}^{n} z_i(0), \tag{22}$$

with a convergence rate equal to:

$$\kappa = \lambda_2(\mathcal{G}) \tag{23}$$

Convergence Rate - Proof - I

- Let us define the disagreement vector as

$$\delta(t) = z(t) - \alpha\,\mathbf{1} \tag{24}$$

whose dynamic is

$$\dot{\delta}(t) = -L\delta(t) \tag{25}$$

- Let us consider the following Lyapunov Function:

$$V(t) = \frac{1}{2}\|\delta(t)\|^2 \tag{26}$$

- Let us consider the derivative:

$$\dot{V}(t) = \delta(t)^T\dot{\delta}(t) = -\delta(t)^T\,L\,\delta(t) \tag{27}$$

Convergence Rate - Proof - II

- Let us recall the inequality:

$$\delta(t)^T L\,\delta(t) \geq \lambda_2(\mathcal{G})\,\|\delta(t)\|^2, \qquad \forall\,\delta(t)\ :\ \mathbf{1}^T\delta(t) = \mathbf{0} \quad (28)$$

- By substituting within the derivative:

$$\dot{V}(t) = -\delta(t)^T L\,\delta(t) \leq -\lambda_2(\mathcal{G})\,\|\delta(t)\|^2 \quad (29)$$

- Thus, denoting $\kappa = \lambda_2(\mathcal{G})$:

$$\dot{V}(t) \leq -2\,\kappa V(t) < 0 \quad (30)$$

Convergence Rate - Proof - III

- From the exponential stability property we have:

$$\|\delta(t)\| \leq M\, e^{-\kappa\, t} \|\delta(0)\|. \tag{31}$$

with M an opportune constant.

- This implies that:

$$\|\delta(t)\| \rightarrow 0, \quad t \rightarrow \infty \tag{32}$$

It follows that:

$$z(t) \rightarrow \alpha\, \mathbf{1}, \quad t \rightarrow \infty \tag{33}$$

- The coefficient α can be determined by applying the left eigenvector property to the null eigenvalue:

$$v_i^T z(t) = e^{-\lambda_i t} v_i^T z(0) \tag{34}$$

Convergence Rate - Proof - IV

- Recall that for $\lambda_1 = 0$ the left eigenvector is $v_1 = \mathbf{1}$, thus:

$$\mathbf{1}^T z(t) = \mathbf{1}^T z(0) \tag{35}$$

as $t \to \infty$ this becomes:

$$\begin{aligned} \mathbf{1}^T \alpha \mathbf{1} &= \mathbf{1}^T z(0) \\ \alpha\, n &= \sum_{i=1}^{n} z_i(0) \\ \alpha &= \frac{1}{n} \sum_{i=1}^{n} z_i(0) \end{aligned} \tag{36}$$

Consensus Problem

Discrete-Time Version

Problem Statement

An iterative version of the consensus algorithm can be stated as follows in discrete-time:

$$z_i(k+1) = z_i(k) + \epsilon \sum_{j \in \mathcal{N}_i} \big(z_j(k) - z_i(k)\big) \tag{37}$$

The discrete-time collective dynamics of the network can be written as:

$$z(k+1) = P\, z(k) \tag{38}$$

where

$$P = I - \epsilon\, L \tag{39}$$

is the Perron matrix of the graph $\mathcal{G}$ with parameter ϵ.

Special Non-negative Matrices

Three different non-negative matrices are of interest for the analysis of the discrete-time consensus algorithm:

- Irreducible: A (non-negative) matrix is reducible if and only if its associated digraph is not strongly connected.
- Stochastic: A (non-negative) matrix is called row (or column) stochastic if all of its row-sums (or column-sums) are $\mathbf{1}$.
- Primitive: An irreducible stochastic matrix is called primitive if it has only 1 eigenvalue with maximum modulus.

Note:
In the case of an undirected graph, then a matrix is irreducible i.f.f. the graph is connected.

Perron-Frobenius Theorem

Let us consider a primitive non-negative matrix P whose left eigenvector w and right eigenvector v satisfy the following conditions:

$$\begin{aligned} Pv &= v, && (40) \\ w^T P &= w^T, && (41) \\ v^T w &= 1. && (42) \end{aligned}$$

Then, the following holds:

$$\lim_{k \to \infty} P^k = v\, w^T. \tag{43}$$

Main Result

Theorem: Consider the collective dynamics given in eq. (38), if the graph is connected and $0 < \epsilon < 1/\Delta_{\text{max}}$, then a consensus is asymptotically achieved to the average of the initial state of all nodes, that is:

$$z(\infty) = \alpha \cdot \mathbf{1}, \tag{44}$$

with:

$$\alpha = \frac{1}{n}\sum_{i=1}^{n} z_i(0), \tag{45}$$

with $z_i(0)$ the initial state of agent i.

Main Result - Proof Sketch I

To prove the theorem, we must prove that:

- P is a primitive matrix with only 1 eigenvalue of modulus 1.
- The (orthonormal) left and right eigenvectors of the matrix P are respectively $v = \mathbf{1}$ and $w = \left(\frac{1}{n}\right)\mathbf{1}$

Then, the result follows from the Perron-Frobenius Theorem:

$$\begin{aligned} \lim_{k\to\infty} x(k) &= v\, w^T x(0) \\ &= \mathbf{1}\left(\frac{1}{n}\right)\left(\mathbf{1}^T x(0)\right) \\ &= \underbrace{\frac{1}{n}\sum_{i=1}^{n} x_i(0)}_{\alpha}\ \mathbf{1} \end{aligned} \tag{46}$$

Main Result - Proof Sketch II

Let us show that P is primitive:

- Irreducible: It follows from the connectivity of the graph $\mathcal{G}$
- Stochastic: It follows from the structure of the Perron matrix:

$$P\,\mathbf{1} = (I - \epsilon L)\,\mathbf{1} = \mathbf{1} \tag{47}$$

- $\exists !\ \rho(P) = 1$: It follows from the fact that:
  - The eigenvalues of L are:

$$\lambda_i \leq \Delta_{\max},\ \forall\, \lambda_i \in \sigma(L)$$

  - The coefficient ϵ is

$$0 < \epsilon < 1/\Delta_{\max}$$

  - Then the eigenvalues of P are:

$$0 < 1 - \epsilon\,\lambda_i \leq 1,\ \forall\, \mu_i \in \sigma(P)$$

Main Result - Proof Sketch III

For the right eigenvector w associated to the eigenvalue $\rho(P) = 1$ we have that:

$$P\mathbf{1} = 1\,\mathbf{1} \tag{48}$$

For the left eigenvector v associated to the eigenvalue $\rho(P) = 1$ we know that P is symmetric, thus:

$$\mathbf{1}^T P = 1\,\mathbf{1}^T \tag{49}$$

By imposing the orthonormality condition it follows that:

$$\begin{aligned} w^T v &= 1 \\ (\beta\,\mathbf{1})^T\,\mathbf{1} &= 1 \\ \beta\,n &= 1 \\ \beta &= \frac{1}{n} \end{aligned} \tag{50}$$

Consensus Problem

Further Results

Overview

These basic results have been extended in several directions:

- Switching Network Topologies
- Directed Network Topologies
- Time-Delays Communication
- Higher-Order Dynamics
- Finite-Time Consensus protocols
- Sample-Data Framework

Reference:

- Y. Cao, W. Yu, W. Ren, and G. Chen. "An Overview of Recent Progress in the Study of Distributed Multi-agent Coordination". In: *ArXiv e-prints* (Sept. 2012). arXiv:1207.3231v2 [math.OC]