Graph Signal Processing and Stationary Graph Signals

Benjamin Girault

Signal and Image Processing Institute
University of Southern California, CA 90089 Los Angeles, USA
benjamin.girault@usc.edu

Objectives:

- **Study signals supported by arbitrary discrete structures rather than Euclidean structures.**
- **Define a mathematical model for stochastic graph signals with a good spectral interpretation.**

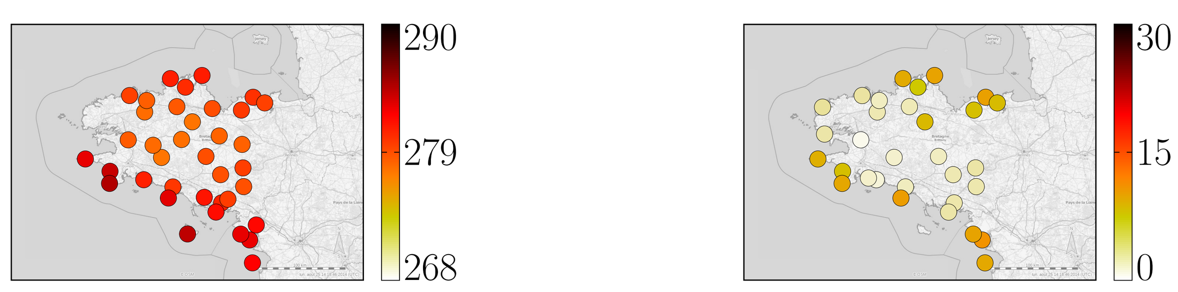
Graph Signal Processing (1, 7)

Basic definitions and results of the field.

Graphs and Signals

Graph, Vertices, Edges, Edge weights: $\mathcal{G} = (V, E, w)$

Signal, Sample: $x\colon V \to \mathbb{R}$ or $\mathbb{C}$, $i \longmapsto x_i$



(a) Temperature (°K) **(b)** Wind Speed (m/s)

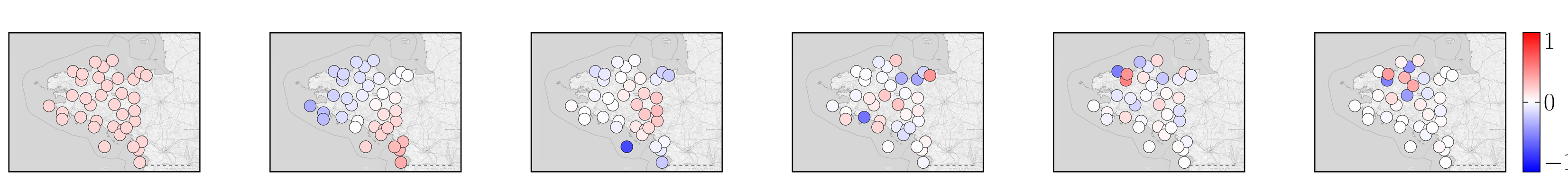
Figure 1: *Geographical graph with weather data. Weights are chosen as a Gaussian kernel of the distance between stations.*

Algebraic Representations of $\mathcal{G}$

Adjacency mat.: $A_{ij} = \begin{cases} w(ij) & \text{if } ij \in E \\ 0 & \text{o.w.} \end{cases}$ Laplacian mat.: $L_{ij} = \begin{cases} \sum_j w(ij) & \text{if } i = j \\ -w_{ij} & \text{o.w.} \end{cases}$

Graph Fourier Transform

Leverage the positive semi-definite property of L to define graph Fourier modes and graph frequencies $\nu_l = \pi\sqrt{\frac{\lambda_l}{\rho_{\mathcal{G}}}}$: $L\chi_l = \lambda_l \chi_l$



(a) $l = 0$ **(b)** $l = 1$ **(c)** $l = 5$ **(d)** $l = 15$ **(e)** $l = 20$ **(f)** $l = 30$

Figure 2: *Several Fourier modes of the graph of Figure 1(a).*

Definition 1 (Graph Fourier Transform).

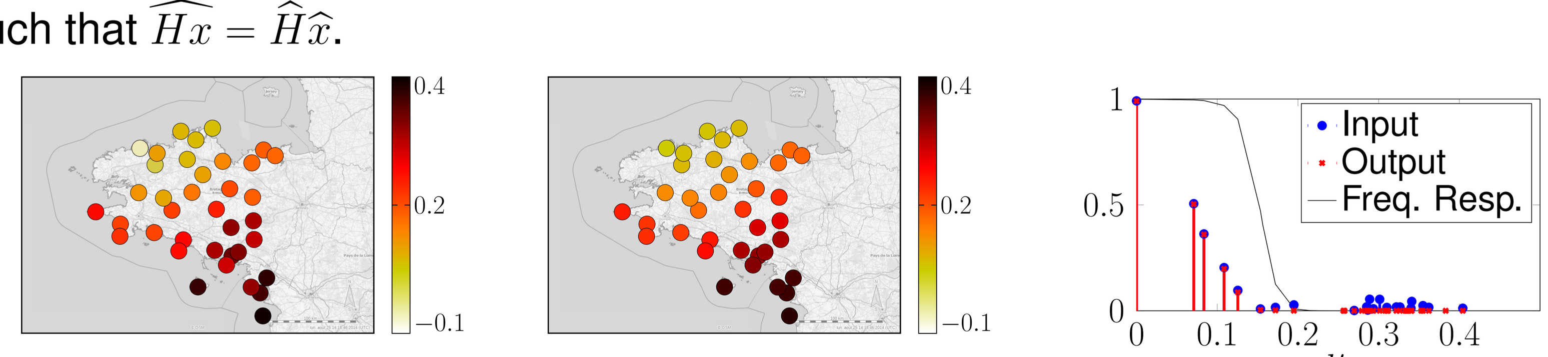
$$\widehat{x}(l) = \langle x, \chi_l \rangle = \sum_{i \in V} x_i \chi_l^*(i) \qquad x_i = \sum_l \widehat{x}(l)\chi_l(i)$$

Filtering

A convolutive operator H verifies the Convolution theorem:

$$H\chi_l = \widehat{h}(l)\chi_l \quad \text{and} \quad \widehat{H} = \operatorname{diag}(\widehat{h}(0), \widehat{h}(1), \dots, \widehat{h}(N-1))$$

such that $\widehat{Hx} = \widehat{H}\widehat{x}$.



(a) Input **(b)** Output **(c)** Fourier Domain

Figure 3: *Example of graph signal filtering.*

Stationary Graph Signals (1)

Question: Model for Stochastic graph signals?

Formal Definition (2)

Statistical Invariance through the Graph Translation operator [3].

Definition 2 (Stationarity). ***Strict:*** $\mathbf{x} \overset{d}{=} T_{\mathcal{G}}\mathbf{x}$, ***Wide:*** $\begin{cases} \mathbb{E}[\mathbf{x}] =: \mu_{\mathbf{x}} = \mu_{T_{\mathcal{G}}\mathbf{x}} \\ \mathbb{E}[\mathbf{x}\mathbf{x}^*] =: R_{\mathbf{x}} = R_{T_{\mathcal{G}}\mathbf{x}} \end{cases}$

Graph Translation (3)

Defined by analogy with the Time Shift as a *phase shift operator on the Fourier components*:

$$\widehat{T_{\mathcal{G}}\chi_l} = e^{-\jmath\nu_l}\chi_l \qquad \text{such that} \qquad T_{\mathcal{G}} = exp\left(-\jmath\pi\sqrt{\frac{L}{\rho_{\mathcal{G}}}}\right)$$

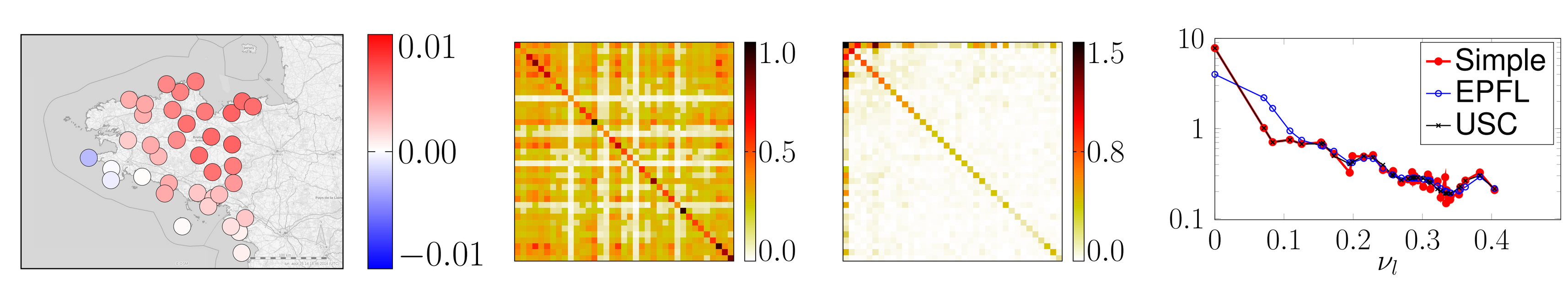
Interpretation as a diffusion operator: Spatially bounded impulse response [4].

WSS Spectral Characterization (2)

$\begin{cases} \widehat{\mu_{\mathbf{x}}} = |\widehat{\mu_{\mathbf{x}}}(0)|\delta \\ S_{\mathbf{x}} := \mathbb{E}[\widehat{\mathbf{x}}\widehat{\mathbf{x}}^*] \text{ diagonal} \end{cases}$ **in other words:** $\begin{cases} \mu_{\mathbf{x}} \text{ is a DC signal} \\ \text{Fourier components are uncorrelated} \end{cases}$

Stationarity of Temperature Increments (5)

Underlying question: Can we consider temperature increments at given time instants as realizations of a stationary graph signal?



(a) μ **(b)** R **(c)** S **(d)** PSD

Figure 4: *Stationarity assessment of temperature increments.*

With a simple estimator of the PSD [2], the EPFL estimator [6], and our estimator [5]

Local Stationarity (5)

Using a wavelet decomposition on graphs to obtain a local notion of PSD, with frequencies replaced by scales.

Objective: Define a notion of local stationarity and find local interpretations of (global) non-stationarity.

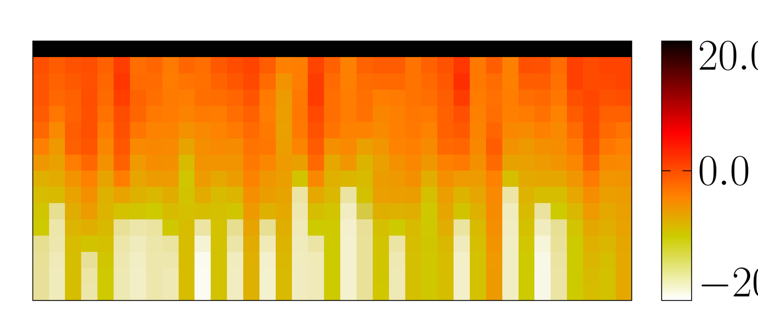


Figure 5: *Local PSD of the temp. incr. (dB).*

Conclusion:

- **Sound definition of stationary graph signals with a good spectral interpretation.**
- **Evidence of stationarity within a weather dataset.**

References

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[2] B. Girault. Stationary Graph Signals using an Isometric Graph Translation. In *Signal Processing Conference (EUSIPCO), 2015 Proceedings of the 23rd European*. IEEE, 2015.

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[4] B. Girault, P. Gonçalves, S. S. Narayanan, and A. Ortega. Localization Bounds for the Graph Translation. In *2016 IEEE GlobalSIP*, Washington D.C., USA, Dec 2016. accepted.

[5] B. Girault, S. S. Narayanan, and A. Ortega. Towards a Definition of Local Stationarity for Graph Signals. submitted to ICASSP 2017.

[6] N. Perraudin and P. Vandergheynst. Stationary signal processing on graphs. *arXiv preprint arXiv:1601.02522*, 2016.

[7] D. I. Shuman, S. K. Narang, P. Frossard, A. Ortega, and P. Vandergheynst. The Emerging Field of Signal Processing on Graphs: Extending High-Dimensional Data Analysis to Networks and Other Irregular Domains. *IEEE Signal Processing Magazine*, 30(3):83–98, 2013.

Signals and Graphs processed using the GraSP toolbox (http://gforge.inria.fr/projects/grasp)