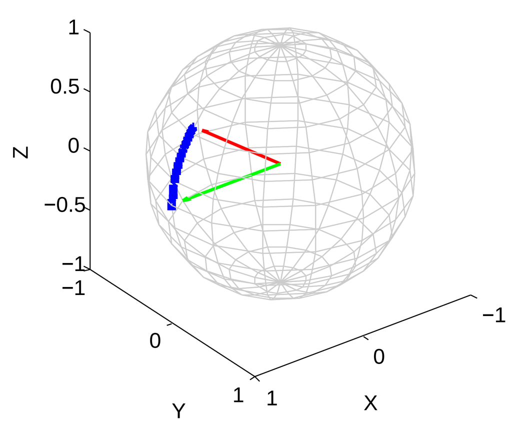
Ming-Hsieh Department of Electrical Engineering

Algebraic structures of linearly controlled Hamiltonians and continuous measurements

Jan Florjanczyk, Todd A. Brun
Communication Sciences Institute

USC Viterbi School of Engineering

General and continuous quantum measurements



- Finite-dimensional quantum systems are represented by vectors in a Hilbert space $|\psi\rangle$
- Quantum states have continuous-time dynamics given by the Schrödinger equation $i\hbar\partial_t|\psi\rangle = H|\psi\rangle$ where H is a *Hamiltonian*
- General quantum measurements can disturb quantum states discontinuously and are defined by a set of measurement operators $\{M_k\}$ with $\sum_k M_k^\dagger M_k = I$ and

$$|\psi\rangle \xrightarrow{\text{prob. } p_k} \frac{M_k|\psi\rangle}{p_k} \qquad \text{where } p_k = \langle\psi|M_k^\dagger M_k|\psi\rangle$$

Why study continuous quantum measurements?

Theoretical consideration

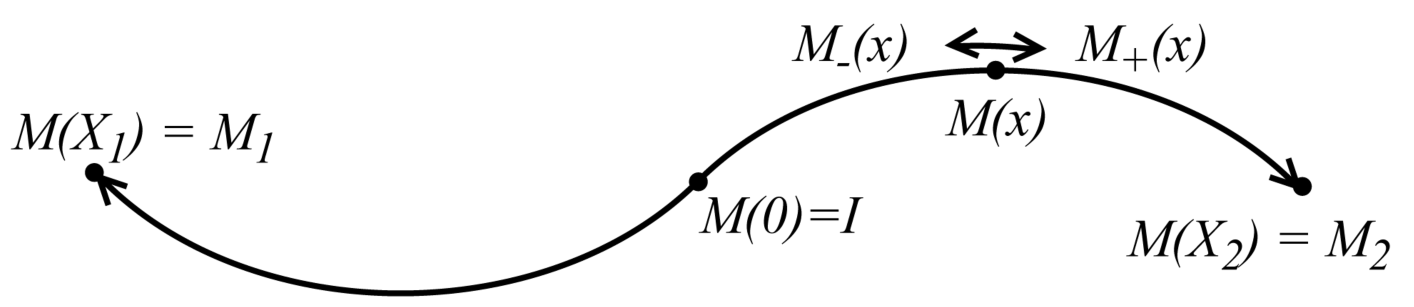
Quantum dynamics are reversible, deterministic and continuous. However, quantum measurements are irreversible, non-deterministic, and discontinuous. Can we describe measurements continuously but allow for irreversible, non-deterministic outcomes?

Experimental considerations

Many quantum mechanical systems either have naturally slow measurement times, or can only be probed weakly. For example: microwave cavities [BHL+90], homo- and heterodyne measurements in quantum optics [YIM86, SSH+87]

Continuous decompositions as random walks

In [OB05] it was shown that any quantum measurement $\{M_1, M_2\}$ can be decomposed into a 1-dimensional continuous stochastic process



In this scheme successive weak measurements (steps) $M_\pm(x)$ are applied at each time-step. These step operators are a function of the running total of measurement outcomes, the pointer variable x. This process can be seen as a 1-d random walk on a curve in operator space. The total walk operator $M(x)$ describes the evolution which terminates at the desired operators M_1 or M_2.

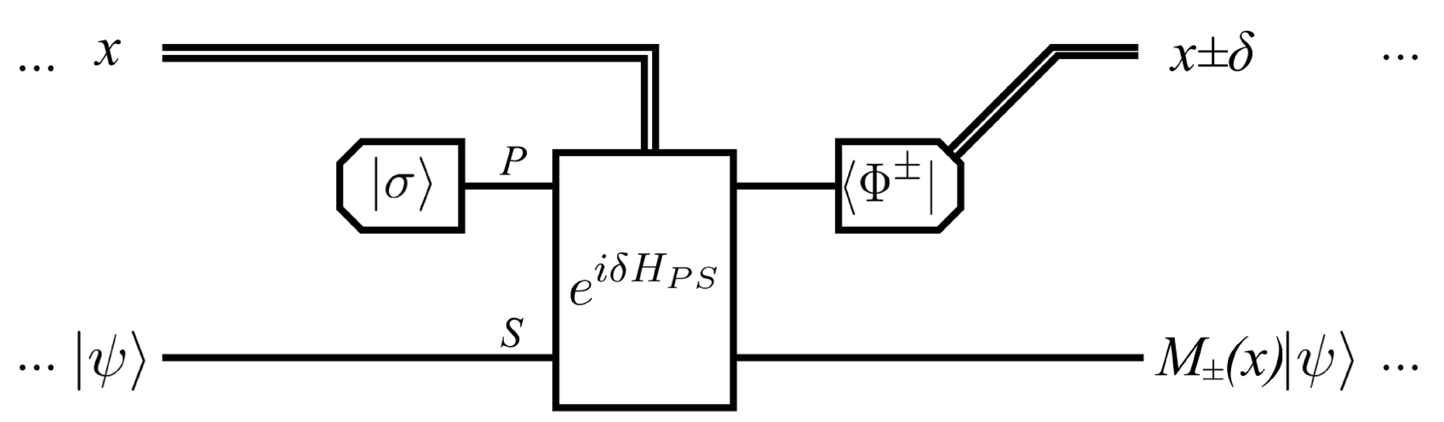
Total walk operator	$M(x) \propto \lim_{\delta\to 0} \prod_{j=1}^{\lfloor \lvert x\rvert/\delta \rfloor} M_\pm(\pm j\delta)$
Endpoint operators	$M_{1,2} = \lim_{x\to X_{1,2}} M(x)$
Reversibility equation	$M_\mp(x\pm\delta)M_\pm(x) \propto \mathbb{1}$

In [FB14] we've shown that any qubit-probe interacting in a fixed way with the system being measured can only decompose measurements of the form

$$M_1 = U_1\left(\alpha\Pi_S + \beta\Pi_{S^\perp}\right)V$$
$$M_2 = U_2\left(\sqrt{1-\alpha^2}\Pi_S + \sqrt{1-\beta^2}\Pi_{S^\perp}\right)V$$

Continuous decomposition with sequences of probes

New question: What can be done with a qubit-probe and a tunable interaction Hamiltonian?



In the above circuit we define the probe state $|\sigma(x)\rangle = |0\rangle$, the detector state $\langle\phi^\pm| = \langle\pm|$ and the tunable interaction

$$H_{PS} = Y_P \otimes \hat\varepsilon(x) \qquad \hat\varepsilon(x) = \sum_{i=0}^{d} p_i(x)H_i$$

Step operators	$M_\pm(x) = \frac{1}{\sqrt{2}}\mathbb{1} + i\delta\langle\pm\lvert H_{PS}(x)\rvert 0\rangle$
Total walk operator	$M(x) = \sum_{i=1}^{D} a_i(x)H_i$

In addition to the reversibility equation, we also introduce

Operator propagation	$\partial_x M(x) = -\hat\varepsilon(x)M(x)$

Quadratic systems of ODEs over non-associative algebras

The reversibility equation and the operator propagation equation can be rewritten as quadratic systems of ODEs

Quadratic ODE sys. (1)	$\sum_{k=0}^{d}\partial_x p_k(x)H_k = \frac{1}{2}\sum_{i,j=0}^{d} p_i(x)p_j(x)\{H_i, H_j\}$
Quadratic ODE sys. (2)	$\sum_{k=0}^{d}\partial_x a_k(x)H_k = -\frac{1}{2}\sum_{i,j=0}^{d} p_i(x)a_j(x)H_iH_j$

Closure lemma

In order to satisfy the reversibility equation, the span of linear control terms

$$F = \text{span}\{H_i\}$$

must be closed under anti-commutation

$$H_i \circ H_j := H_iH_j + H_jH_i \in F \qquad \forall\, H_i, H_j \in F.$$

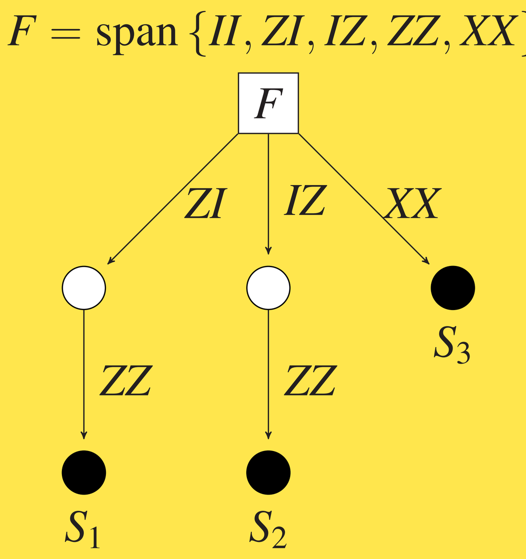
Restriction algorithm

function RESTRICT(F)
  if $F^{\circ 2} = F$ **then return** F
  else
    $S \leftarrow$ BASIS$\left(F^{\circ 2}\right)$
    for all $s \in S$ **do**
      $s \leftarrow$ RESTRICT$\left(F \setminus s\right)$
    end for
    return S
  end if
end function

In the example, we show the branching calls to RESTRICT(F). At each branch, we show which control term is dropped. The algorithm terminates on the filled nodes.

Example

INPUT: $F = \text{span}\{II, ZI, IZ, ZZ, XX\}$



RESTRICT$(F) = \{S_1, S_2, S_3\}$

$$S_1 = \{II, IZ, XX\}$$
$$S_2 = \{II, ZI, XX\}$$
$$S_3 = \{II, ZI, IZ, ZZ\}.$$

Extension algorithm

function EXTEND(F)
  while $F^{\circ 2} \supset F$ **do**
    $F \leftarrow$ BASIS$\left(F^{\circ 2}\right)$
  end while
end function

Example

INPUT: $F = \text{span}\{II, ZI, IZ, ZZ, XX\}$
EXTEND$(F) = \text{span}\{II, ZI, IZ, ZZ, XX, YY\}$
In this example we see that the only additional control term needed to complete the algebra was YY

Discussion

- Using the control set $F = \text{span}\{I, X, Z\}$ we can only decompose measurements of the form achieved by qubit-probe feedback above.
- Quadratic ODE system (2) is completely determined by system (1) and the initial condition $M(0) = I$.
- Quadratic ODE system (1) contains no orbits [KS95].
- The performance of RESTRICT(F) is sub-optimal, this is due to the freedom of choice for BASIS$(F^{\circ 2})$
- If the span of controls F is also closed under

$$H_1H_2H_3H_4 + H_4H_3H_2H_1 \in F$$

then by the *Cohn Reversible Theorem* F is the Hermitian part of the Free algebra generated by F (i.e.: the most general algebra). [McC04]

Acknowledgements JF and TAB thank Daniel Lidar and Ognyan Oreshkov for useful discussions. This research was supported in part by the ARO MURI under Grant No. W911NF-11-1-0268.

References

- M. Brune, S. Haroche, V. Lefevre, J. M. Raimond, and N. Zagury. Quantum nondemolition measurement of small photon numbers by rydberg-atom phase-sensitive detection. *Phys. Rev. Lett.*, 65:976–979, Aug 1990.
- Jan Florjanczyk and Todd A. Brun. Continuous decomposition of quantum measurements via qubit-probe feedback. *Phys. Rev. A*, 90:032102, Sep 2014.
- M.K. Kinyon and A.A. Sagle. Quadratic dynamical systems and algebras. *Journal of Differential Equations*, 117(1):67 – 126, 1995.
- Kevin McCrimmon. *A taste of Jordan algebras.* Springer, 2004.
- O. Oreshkov and T.A. Brun. Weak measurements are universal. *Physical Review Letters*, 95(11):110409, 2005.
- J. H. Shapiro, G. Saplakoglu, S.-T. Ho, P. Kumar, B. E. A. Saleh, and M. C. Teich. Theory of light detection in the presence of feedback. *J. Opt. Soc. Am. B*, 4(10):1604–1620, Oct 1987.
- Y. Yamamoto, N. Imoto, and S. Machida. Amplitude squeezing in a semiconductor laser using quantum nondemolition measurement and negative feedback. *Phys. Rev. A*, 33:3243–3261, May 1986.