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On Constructing Good Compressed Sensing Matrices

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Why Deterministic Constructions?

Basis Pursuit (BP) Recovery

$$\begin{aligned}&\underset{\underline{x}\in\mathbb{R}^n}{\text{minimize}} \quad \|\underline{x}\|_1\\ &\text{subject to } \underline{y} = \mathbf{A}\underline{x}\end{aligned}$$

- Checking RIP and other recovery criteria on $\mathbf{A}$ is NP-hard
- Purely random matrices computationally expensive in BP

Design Parameters

BP constraint

$$\underline{y} = \mathbf{A}\underline{x} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathsf{T}}\underline{x}$$

- $\mathbf{U}$ plays no role
- $\mathbf{V}$ is the most important design parameter
- Role of $\mathbf{\Sigma}$ limited and depends on $\mathbf{V}$

Recovery Guarantees in Literature

1. By Coherence $\mu(\mathbf{A})$ (Donoho-Elad-03)
   If $\underline{x}$ satisfies $\|\underline{x}\|_0 < 0.5\left(1+\mu(\mathbf{A})^{-1}\right)$ then BP succeeds in exact recovery
2. Null Space Property (Zhang-08)
   If $\forall\,|S| \leq k$ and $\forall \underline{d} \in \mathcal{N}(\mathbf{A})\setminus\{0\}$, $\|\underline{d}_S\|_1 < \|\underline{d}_{S^c}\|_1$ then BP recovers correctly all k-sparse vectors $\underline{x}$

Prior Art

Algebraic Constructions

- (DeVore-07) Construct RIP matrices using polynomials over finite fields
- (Calderbank-Howard-Jafarpour-10) Demonstrate statistical RIP for linear coding matrices
- (Jafarpour-Xu-Hassibi-Calderbank-09) Expander graph based construction for compressed sensing matrices

Optimization based Constructions

- (Tropp-Dhillon-Heath-Strohmer-05) Grassmannian frame design using alternating projection

We shall take an optimization based approach as well

Minimizing Coherence $\mu(\mathbf{A})$

Optimization Problem

$$\begin{aligned}&\underset{\mathbf{X}\in\mathbb{R}^{n\times n}}{\text{minimize}} \quad \max_{i<j}|x_{ij}|\\ &\text{subject to } x_{kk}=1, \quad 1\leq k\leq n\\ &\qquad \mathbf{X}\succeq \mathbf{0}\\ &\qquad \text{rank}(\mathbf{X})\leq m\end{aligned}$$

- Optimization variable is the Gram matrix $\mathbf{A}^{\mathsf{T}}\mathbf{A}$, factorization yields $\mathbf{A}$
- Hard problem! - Rank constraint with a constant trace
- Much simpler if $\mathcal{R}(\mathbf{A})$ is (partially) known

Constructing $\mathcal{R}(\mathbf{A})$ I

Initialization

- Measurements incoherent with sparsity basis are good
- Hadamard matrices over $\mathbb{R}^{n\times n}$ maximally incoherent
- Use $1+\log_2 n$ rows of a Hadamard matrix in $\mathbf{A}$
- Optimal decoder needs atleast $1+\log_2 n$ measurements

Example: $n=8$

$$\mathbf{A} = \begin{bmatrix}1&1&1&1&1&1&1&1\\1&1&1&1&-1&-1&-1&-1\\1&1&-1&-1&1&1&-1&-1\\1&-1&1&-1&1&-1&1&-1\end{bmatrix}$$

Constructing $\mathcal{R}(\mathbf{A})$ II

Iteratively growing $\mathcal{R}(\mathbf{A})$ using $\mathcal{N}(\mathbf{A})$

1. Solve the following optimization problem to get $\underline{z}_j^\star$

$$\begin{aligned}&\underset{\underline{z}\in\mathbb{R}^n, z_j=1}{\text{minimize}} \quad \|\underline{z}\|_1\\ &\text{subject to } \mathbf{A}\underline{z} = \underline{0}\end{aligned}$$

2. Compute for each $1\leq j\leq n$

$$f_j = \frac{\max_{|S|\leq k}\left\|\left(\underline{z}_j^\star\right)_S\right\|_1}{\left\|\underline{z}_j^\star\right\|_1}$$

3. Append $\left(\underline{z}_{j_0}^\star\right)^{\mathsf{T}}$ to $\mathbf{A}$ if $f_{j_0}\geq 0.5$, where $j_0 = \arg\max_{1\leq j\leq n} f_j$

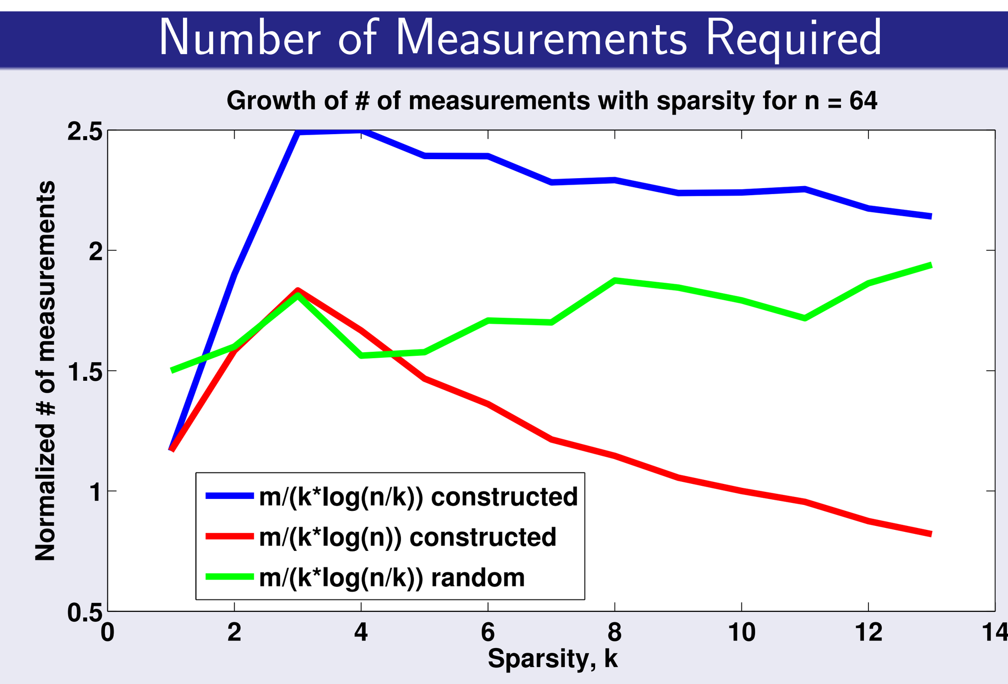
Minimize $\mu(\mathbf{A})$ given $\mathcal{R}(\mathbf{A})$

Easier (Convex) Optimization Problem

$$\begin{aligned}&\underset{\mathbf{X}\in\mathbb{R}^{n\times n}}{\text{minimize}} \quad \max_{i<j}|x_{ij}|\\ &\text{subject to } x_{kk}=1, \quad 1\leq k\leq n\\ &\qquad \mathbf{X}\succeq \mathbf{0}\\ &\qquad \mathcal{R}(\mathbf{X}) = \mathcal{R}(\mathbf{A})\end{aligned}$$

- Computationally most expensive step on general purpose interior point solver SeDuMi
- CVX unable to handle dimensions $n > 100$
- Is this step really necessary?

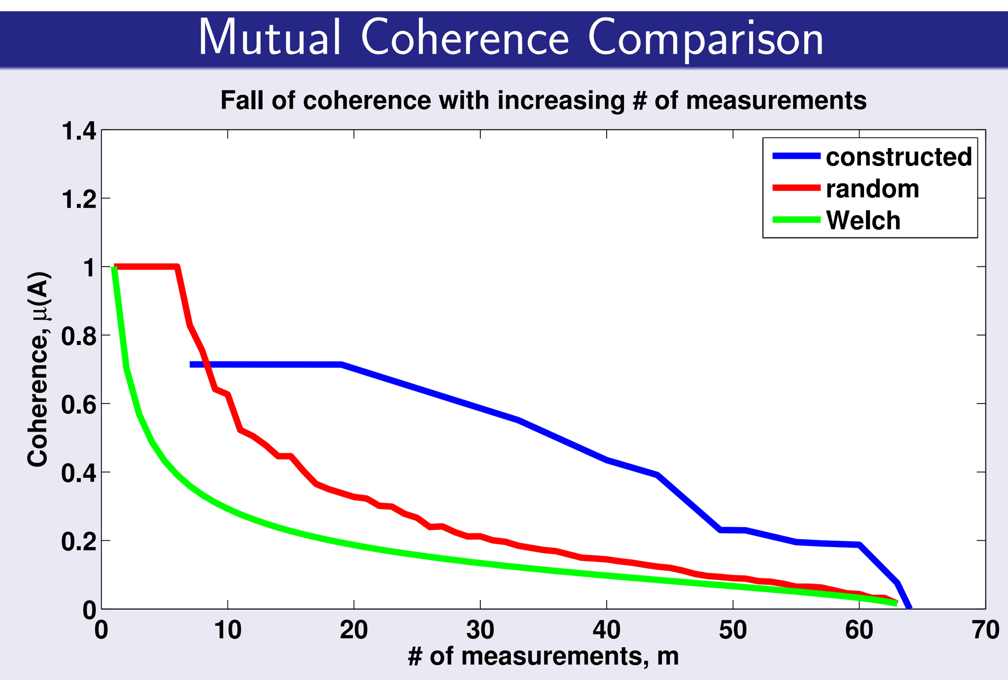
Number of Measurements Required



Observation

- Optimal scaling of m achieved to within a constant factor of 2 as compared to random subsampling of Hadamard matrix

Mutual Coherence Comparison



Observation

- Non-random construction does not have as good mutual coherence performance as random Hadamard subsampling

Summary

Contribution

- Developed a non-random, optimization based construction for minimum mutual coherence sensing matrix which achieves optimal scaling of number of measurements with sparsity

Future Work

- Is it possible to grow a sensing matrix using this approach to satisfy RIP given an initialization satisfying RIP of order 2? (Open Problem)

References

- Y. Zhang, "Theory of compressive sensing via l1-minimization: a non-rip analysis and extensions," *Rice University CAAM Technical Report TR08–11*, pp. 19–22, 2008.
- J. A. Tropp, I. S. Dhillon, R. W. Heath, Jr., and T. Strohmer, "Designing structured tight frames via an alternating projection method," *IEEE Trans. Inform. Theory*, vol. 51, no. 1, pp. 188–209, 2005.

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